

Some Remarks on the Geodesic Completeness of Compact Nonpositively Curved Spaces

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Let X be a geodesic space. We say that X is geodesically complete if every geodesic segment $\beta : [0, a] \rightarrow X$ from $\beta(0)$ to $\beta(a)$ can be extended to a geodesic ray $\alpha : [0, \infty) \rightarrow X$, (i.e. $\beta(t) = \alpha(t)$, for $0 \leq t \leq a$).

If X is a compact npc space (“npc” means: “non-positively curved”) then it is *almost geodesically complete*, see [10]. (X , with metric d , is almost geodesically complete if its universal cover \tilde{X} satisfies the following property: there is a constant C such that for every $p, q \in \tilde{X}$ there is a geodesic ray $\alpha : [0, \infty) \rightarrow \tilde{X}$, $\alpha(0) = p$, and $d(q, \alpha) \leq C$.)

Then it is natural to ask if in fact every compact npc space has some kind of geodesically complete npc core. In view of this, we ask the following question: (this question was already stated in [1], p. 4)

Question. Does a compact npc geodesic space Z have a subspace A , such that the inclusion $A \rightarrow Z$ is a homotopy equivalence and A , with the intrinsic metric, is npc and geodesically complete?

Remark. One could also ask whether there is such an A that is also totally geodesic in Z , but this is certainly false. Just take Z to be a flat 2-torus minus an open disk. Then the only possibility for A is a figure eight that can not be totally geodesic in Z .

If Z is piecewise smooth npc 2-complex, then, by collapsing the free faces, we get a subcomplex which is, with its intrinsic metric, geodesically complete and npc (see [2], and also [5], p.208). Hence the answer to the problem above is YES, when Z is piecewise smooth a 2-complex.

For the general case the answer is NO. To show this, in section 2 we construct a space Z with the following properties:

First Example. Z is a finite piecewise flat npc 3-complex with the property that there is no PL subspace A of Z such that

- (i) The inclusion $\iota : A \rightarrow Z$ is a homotopy equivalence.
- (ii) A , with the intrinsic metric determined by the metric of Z , is npc geodesically complete.

But the space Z we construct in this example *does have* a subspace A satisfying (i) and also (ii) but with other geodesic metric (not the intrinsic one). Hence we are forced to ask a deeper question.

Question. Does a compact npc geodesic space Z have a subspace A , such that the inclusion $A \rightarrow Z$ is a homotopy equivalence and A admits a geodesic metric that is npc and geodesically complete?

For this question the answer is also NO. To show this, in section 3 we construct an space Z with the following properties:

Second Example. *Z is a finite piecewise flat npc 3-complex with the property that there is no PL subspace A of Z such that*

- (i) *The inclusion $\iota : A \rightarrow Z$ is a homotopy equivalence.*
- (ii) *A admits a npc geodesically complete geodesic metric.*

Of course the second example is also a counterexample to the first question but, even though it is a little non-standard to present these counterexamples in this way, we think it is interesting to show how the constructions evolve.

Still, the space Z we construct for theorem E is homotopically equivalent to a npc geodesically complete space. Hence we are again forced to ask an even deeper question, for which we do not know the answer.

Open Question. Is a compact npc geodesic space Z homotopically equivalent to a compact npc geodesically complete geodesic space Z' ?

We say that a group is npc if it is isomorphic to the fundamental group of a compact npc geodesic space, and a group is gc-npc (i.e. geodesically complete non-positively curved) if it is isomorphic to the fundamental group of a compact npc geodesically complete geodesic space. With these definitions, we restate the open problem above in the following way: Is every npc group a gc-npc group? Remark that if we drop the compactness condition the answer is always YES, at least in the PL category: let K be a piecewise flat npc simplicial complex. Embed it in some \mathbb{R}^N and let Y be the hyperbolization of \mathbb{R}^N relative to K . Y can be given a complete npc metric having K as a totally geodesic subspace (see [8]). Let Z be the cover of Y relative to the subgroup $\pi_1(K)$ of $\pi_1(Y)$. Then Z is a non-compact npc geodesically complete space having K as a deformation retract.

For a group Γ define the *min-gc-dimension* and the *max-gc-dimension* in the following way. *min-gc-dim* Γ is the minimum of the n 's such that there is a compact npc geodesically complete n -complex with fundamental group isomorphic to Γ . If there is not such complex we put *min-gc-dim* $\Gamma = +\infty$. Similarly *max-gc-dim* Γ is the maximum of the n 's such that there is a compact npc geodesically complete n -complex with fundamental group isomorphic to Γ . If there is not such complex we put *max-gc-dim* $\Gamma = -\infty$, and if there are infinite such n 's we write *max-gc-dim* $\Gamma = +\infty$.

Note that we always have $gd\Gamma \leq \text{min-gc-dim } \Gamma$, provided *min-gc-dim* $\Gamma \neq +\infty$. Here

$gd\Gamma$ is the geometric dimension of Γ . This motivates the following question.

Open Question. Is it true that $\min\text{-}gc\text{-}dim\ \Gamma = gd\ \Gamma$, provided that $\min\text{-}gc\text{-}dim\ \Gamma \neq +\infty$?

In [4] it is given an example of a group Γ , with $gd\ \Gamma = 2$, that is the fundamental group of a npc compact 3-complex but it is not the fundamental group of a npc compact 2-complex. Hence, at least one of the two open problems above have a negative answer.

It is easy to show that for the infinite cyclic group F_1 we have $\min\text{-}gc\text{-}dim\ F_1 = \max\text{-}gc\text{-}dim\ F_1 = gd\ F_1 = cd = F_1 = 1$. Here cd is the cohomological dimension. But for $n > 2$ things are different. (F_n denotes the free group with N generators.)

Proposition 1. $\max\text{-}gc\text{-}dim\ F_n > 1$, for $n \geq 2$.

Because $\min\text{-}gc\text{-}dim\ F_n = 1$, for all n , we have that in general is not true that $\min\text{-}gc\text{-}dim$ and $\max\text{-}gc\text{-}dim$ are equal. And also, in general, it is not true that $\max\text{-}gc\text{-}dim$ and gd are equal. In fact in general it is not true that $\min\text{-}gc\text{-}dim$ and $\max\text{-}gc\text{-}dim$ are equal, even if Γ is the fundamental group of a closed manifold, as the next proposition asserts.

Proposition 2. Let G_n denote the fundamental group of the genus n surface. Then $\max\text{-}gc\text{-}dim\ G_n > 2$, for n large.

In fact this proposition is a corollary of the following proposition.

Proposition 3. For every m there is an k such that $\max\text{-}gc\text{-}dim\ G_n \geq m$, for $n \geq k$.

In fact, using the same method of the proof of Proposition 3 we get that also for F_n we get: For every m there is an k such that $\max\text{-}gc\text{-}dim\ F_n \geq m$, for $n \geq k$.

From the proof of proposition 1, it seems that the answer to the following problem is YES.

Open Question. Is $\max\text{-}gc\text{-}dim\ F_n$ finite?

Or, more generally.

Open Question. Is $\max\text{-}gc\text{-}dim\ \Gamma$ finite, provided $\min\text{-}gc\text{-}dim\ \Gamma \neq +\infty$?

Open Question. Assume that $\min\text{-}gc\text{-}dim\ \Gamma \neq +\infty$, and let $\min\text{-}gc\text{-}dim\ \Gamma \leq m \leq \max\text{-}gc\text{-}dim\ \Gamma$. Does there exist a compact npc geodesically complete m -complex with fundamental group isomorphic to Γ ?

The next proposition says that a npc group can be made geodesically complete after making a free product with some free group, at least in the PL category.

Proposition 4. *Let Γ be the fundamental group of a finite piecewise flat npc simplicial complex. Then there is an n such that $\Gamma * F_n$ is gc-npc.*

The number n in proposition D is very large and it would be interesting to know the relationship between n and Γ . For example for $n = 1$ we can ask:

Open Question. Does $\Gamma * F_1$ gc-npc implies Γ gc-npc.

Finally, it is important to remark that geodesic completeness is a useful property related to rigidity results. In certain cases rigidity results in the manifold category can be generalized replacing the manifold condition by the geodesic completeness condition. See for example the work of Leeb [9], and Davis-Okum-Zheng [7]. One of the simplest cases of this is the topological version of Gromoll-Wolf-Lawson-Yau torus theorem (see [1],[3]). For example, if Z is a compact npc geodesic space homotopically equivalent to some torus \mathbb{T}^n then there is a totally geodesic embedding $\mathbb{T}^n \rightarrow Z$ and easily follows that if Z is geodesically complete, Z is isometric to \mathbb{T}^n . But propositions C and D show that geodesic completeness is, in general, not enough. In fact, the dimension of the spaces could be the same and, still, we do not get rigidity, as the next proposition shows.

Proposition 5. *For $n \geq 2$ there is a finite piecewise flat npc geodesically complete 2-complex Z with $\pi_1(Z) \cong G_n$ and Z is not a 2-manifold.*

Hence, if we want to generalize rigidity results (like Farrell-Jones topological rigidity of npc manifolds) to non-manifold categories, we need more than geodesic completeness. On the other hand the singularities of the Z we construct in proposition F are quite trivial and one wonders whether this is a general fact.

Here is a short outline of the paper. In section 1 we give some definitions and a lemma. In section 2 we construct the first example. In section 3 we construct the second example. In section 4 we prove propositions 1,3,4 and 5.

1. Definitions.

1. Definitions and Lemmas.

Let Δ^n denote the canonical n -simplex (i.e. Δ^n is the convex hull of the $n + 1$ points $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ in \mathbb{R}^{n+1}). Let also \mathbb{T}^n denote the n -torus with its canonical PL structure. Now, let Λ be a PL subspace of \mathbb{T}^3 PL homeomorphic to Δ^2 . We construct the PL space T by taking two copies T^0 and T^1 of \mathbb{T}^3 and identifying them along Λ , and we consider Λ , T^0 and T^1 as a PL subspaces of this quotient space T .

Consider a pair (K, A) , where K is a PL space, $A = \{\sigma_1, \dots, \sigma_r\}$, and each σ_i is a PL subspace of K PL homeomorphic to Δ^2 . Given these data we construct the PL space $\mathcal{T}(K, A)$ by taking, for each σ_i , a copy of T and identifying Λ with σ_i . We consider K , each

copy T_i of T , and each copy T_i^j of \mathbb{T}^j , $j = 0, 1$ as being PL subspaces of $\mathcal{T}(K, A)$. Also, if J is a PL subspace of K with $\sigma_i \subset J$, for all i , we can consider $\mathcal{T}(J, A) \subset \mathcal{T}(K, A)$.

Finally, a free face of a simplicial complex J is a n -simplex σ in J that is the face of exactly one $n + 1$ -simplex in J .

Lemma 1.1. *Let K and A be as above and Y a closed PL subspace of $X = \mathcal{T}(K, A)$ such that the inclusion $\iota : Y \rightarrow X$ is a homotopy equivalence. Then $T_i \subset Y$, $i = 1, \dots, r$.*

proof. Take a point p in $T_i \setminus K$, for some $i = 1, \dots, r$. We have that $H_3(X, X \setminus \{p\}) \cong \mathbb{Z}$. (Here H denotes homology with \mathbb{Z} coefficients.) Also, the map $H_3(X) \rightarrow H_3(X, X \setminus \{p\})$ is onto. Because $\iota_* : H_3(Y) \rightarrow H_3(X)$ is an isomorphism, we get that the composition $H_3(Y) \rightarrow H_3(X) \rightarrow H_3(X, X \setminus \{p\})$ is onto. Hence Y is not contained in $X \setminus \{p\}$. This means that $T_i \setminus K \subset Y$. Hence $T_i = \overline{T_i \setminus K} \subset \overline{Y} = Y$. This proves the lemma.

Remark. Note that lemma 5.1 implies that each σ_i is in Y . Hence we have that $Y = \mathcal{T}(J, A)$, for some closed PL subspace J of K , with $\sigma_i \subset J$, $i = 1, \dots, r$.

Lemma 1.2. *Let K and A be as above and J a PL subspace of K . If the inclusion $\eta : \mathcal{T}(J, A) \rightarrow \mathcal{T}(K, A)$ is a homotopy equivalence, then the inclusion $\iota : J \rightarrow K$ is also a homotopy equivalence.*

proof. We have the following diagram.

$$\begin{array}{ccc} J & \xrightarrow{\iota} & K \\ \alpha \downarrow & & \beta \downarrow \\ \mathcal{T}(J, A) & \xrightarrow{\eta} & \mathcal{T}(K, A) \\ r \downarrow & & s \downarrow \\ J & \xrightarrow{\iota} & K \end{array}$$

Here all the arrows of the upper square are inclusions and the vertical arrows of the lower square are retractions induced by retractions of each T_i to σ_i . Applying the i -th homotopy functor to the diagram above we get

$$\begin{array}{ccc} \pi_i(J) & \xrightarrow{\iota_*} & \pi_i(K) \\ \alpha_* \downarrow & & \beta_* \downarrow \\ \pi_i(\mathcal{T}(J, A)) & \xrightarrow{\eta_*} & \pi_i(\mathcal{T}(K, A)) \\ r_* \downarrow & & s_* \downarrow \\ \pi_i(J) & \xrightarrow{\iota_*} & \pi_i(K) \end{array}$$

Because $r\alpha = id_J$, $s\beta = id_K$ and η_* is, by hypothesis, an isomorphism we get that ι_* is an isomorphism. This proves the lemma.

Lemma 1.3. *Let J be a PL subspace of Δ^3 , $A = \{\sigma_1, \dots, \sigma_r\}$ a set of PL subsets of J , PL equivalent to Δ^2 . Assume that $B = \cup A = \cup_i \sigma_i$ is contractible. If $Y = \mathcal{T}(J, A)$ admits a npc geodesically complete geodesic metric, then J is two dimensional (i.e. PL equivalent to a two complex).*

proof. We can assume that J is a simplicial complex such that each σ_i is a subcomplex of J . Let K be a simplicial complex PL equivalent to Δ^3 , such that we can consider J as being a subcomplex of K . Suppose that J contains at least one 3-simplex. Let $c = \sum c_k$, a \mathbb{Z}_2 simplicial 3-chain in Δ^3 , where the c_i 's are all the 3-simplices of J , without repetition. Because Y is geodesically complete we have that Y does not have free faces (see [10]). Hence, every 2-simplex in J is in the boundary of at least two 3-simplices of Y . But $J \subset \Delta^3$, thus every 2-simplex of J is in the boundary of exactly two 3-simplices, unless the 2-simplex lies in some $\sigma_i \subset B$. Hence $\partial c = 0$, and follows that $H_3(\Delta^3, B) \neq 0$. This is a contradiction because B is contractible, hence $H_3(\Delta^3, B) = 0$. This proves the lemma.

Lemma 1.4. *Assume that $X = \mathcal{T}(K, A)$ admits a npc geodesic metric d . Then K is totally geodesic in X .*

proof. First, by the topological flat torus theorem (see, for example, [3]), it is easy to see that each T_i^j , is totally geodesic, for all i and $j = 0, 1$. Hence $\Lambda_i = T_0^i \cap T_1^i$ is totally geodesic. We prove now that K is also totally geodesic. Let $p, q \in K$, and $\alpha : [0, d(p, q)] \rightarrow X$ a geodesic with $\alpha(0) = p$, and $\alpha(d(p, q)) = q$. For each T_i let $a = \min\{t : \alpha(t) \in T_i\}$ and $b = \max\{t : \alpha(t) \in T_i\}$. Then $\alpha|_{[a, b]}$ is a geodesic joining $\alpha(a)$ to $\alpha(b)$. But $\alpha(a), \alpha(b) \notin T_i \setminus K$ (because is open), hence $\alpha(a), \alpha(b) \in T_i \cap K = \Lambda_i$. But we saw before that Λ_i is totally geodesic. Consequently $\alpha[a, b] \subset \Lambda_i \subset K$. This proves the lemma.

2. First Example.

Consider $\Delta^3 \subset \mathbb{R}^4$ with the flat metric induced by \mathbb{R}^4 . Fix a vertex v of Δ^3 . Let $A = \{\sigma_1, \sigma_2, \sigma_3\}$, where the σ_i 's are the 2-simplices of $\partial\Delta^3$ that contain v . Write $B = \cup A = \sigma_1 \cup \sigma_2 \cup \sigma_3$. Take $X = \mathcal{T}(\Delta^3, A)$. Consider \mathbb{T}^3 with its canonical flat metric and $\Lambda \subset \mathbb{T}^3$ isometric to Δ^2 . Because each σ_i in Δ^3 is isometric to Δ^2 , all the identifications used to construct X are isometries. Hence this determines a piecewise flat npc metric on X .

Proposition 2.1. *There is no PL subspace Y of X such that*

- (i) *The inclusion $\iota : Y \rightarrow X$ is a homotopy equivalence.*
- (ii) *Y , with the intrinsic metric determined by the metric of X , is geodesically complete non-positively curved.*

proof. Suppose such a Y exists. By the remark after lemma 5.1. we have that $Y = \mathcal{T}(J, A)$, for some PL subspace J of Δ^3 , and each $\sigma_i \subset J$. By lemma 5.3, J is two complex. Hence $\text{link}(v, J)$ is one dimensional. Because $\sigma_i \subset J$ and $v \in \sigma_i$, $i = 1, 2, 3$, we get a loop of length $3\frac{\pi}{3} = \pi$ in $\text{link}(v, J)$ (each σ_i determines a path of length $\frac{\pi}{3}$ in $\text{link}(v, J)$). Thus J , with the intrinsic metric, is not npc. But lemma 5.4 says that J is totally geodesic in Y , hence J is npc, a contradiction. This proves the proposition.

If we choose $J = \sigma_1 \cup \sigma_2 \cup \sigma_3$, and $Y = \mathcal{T}(J, A)$, then we can provide this Y with a geodesically complete piecewise flat npc metric: give each σ_i a flat metric in such a way that

the angle at $v \in \sigma_i$ is $\frac{\pi}{3}$ and choose the 2-simplices Λ_i accordingly. (Note that this metric on Y is *not the intrinsic metric*.) With this choice of Y we certainly have that the inclusion $\iota : Y \rightarrow X$ is a homotopy equivalence. Consequently, our space X does admit a PL subspace Y that admits a geodesically complete non-positively curved metric. In our next example this will not happen.

3. Second Example.

Consider again $\Delta^3 \subset \mathbb{R}^4$ with the flat metric induced by \mathbb{R}^4 . Let $L \subset \Delta^3$ be a contractible two complex with no free faces. (Take for example the house with two rooms, see [6].) Let A be the set of all 2-simplices of L and take $X = \mathcal{T}(\Delta^3, A)$. Note that we can give X a npc piecewise flat metric in the following way. Let $\sigma_i \in A$ and choose $\Lambda_i \in T_i$ isometric to σ_i . In this way all the identifications used to construct X are isometries and this determines a piecewise flat npc metric on X .

Proposition 3.1. *There is no PL subspace Y of X such that*

- (i) *The inclusion $\iota : Y \rightarrow X$ is a homotopy equivalence.*
- (ii) *Y admits a geodesically complete non-positively curved geodesic metric.*

proof. Suppose such a Y exists. By the remark after lemma 5.1. we have that $Y = \mathcal{T}(J, A)$, for some PL subspace J of Δ^3 , and that each $\sigma_i \subset J$. Hence we can assume that J is a simplicial complex and that L is a subcomplex of J . By lemmas 5.2 and 5.3, J is a contractible two complex.

claim. *J has no free faces (or, in this case, free edges).*

If J has a free edge e , we have two possibilities. First, if e is not in L then e would also be a free edge of Y , but this is impossible because Y is geodesically complete non-positively curved, thus has no free faces (see [10]). Second, e can not be in L because $L \subset J$ and L , by hypothesis, has no free faces. This proves the claim.

But the claim and lemma 5.4 imply that J is a contractible complex with no free faces that admits a geodesically complete npc geodesic metric. This is impossible (see [10]). This proves the proposition.

So, our X above does not have a subspace that admits a geodesically complete npc geodesic metric. But there is a space Z homotopically equivalent to Y that admits a geodesically complete npc geodesic metric. In fact, X is homotopically equivalent to a finite wedge of tori, which certainly admits such a metric.

4. Proofs of Propositions 1,3,4 and 5.

proof of proposition 1. Let $C = [0, 1] \times \mathbb{S}^1$. Then C is npc and $S_0 = \{0\} \times \mathbb{S}^1$ and $S_1 = \{1\} \times \mathbb{S}^1$ are totally geodesic in C . Let s_0 be an embedded segment of length one in the interior of C . Let $n \geq 3$. Let s_1, \dots, s_{n-2} be embedded segments in the interior of C of length $\frac{1}{n-2}$ such that s_0, s_1, \dots, s_{n-2} are all disjoint. Let u_1, \dots, u_{n-2} be segments of length $\frac{1}{n-2}$ embedded in S_1 with $S_1 = \bigcup u_i$. Because all s_i 's, $i > 0$, and u_i 's are totally geodesic and isometric (they have the same length) we can identify each s_i , $i > 0$, with u_i . Also, because S_0 and s_0 have the same length there is a surjective local isometry $s_0 \rightarrow S_0$, and we can use this local isometry to identify s_0 with S_0 . Let Z be the resulting space. Then Z is npc, compact and it is easy to see that $\pi_1(Z) \cong F_n$. Also the simplicial complex Z has no free faces. Hence it is geodesically complete (see [5], p.208).

For $n = 2$ the construction is similar. Just replace the cylinder $C = [0, 1] \times \mathbb{S}^1$ by a Moebius band. This proves proposition 1.

proof of proposition 4. The proof is similar to the proof of proposition 1. Let Z be a finite piecewise flat npc simplicial complex. Let σ be a free face of dimension l . Then σ is in the boundary of exactly one $(l + 1)$ -simplex. Call this simplex τ . Subdivide σ and let s_1, \dots, s_k be the l -simplices of this subdivision and assume $\text{diam } s_i < \text{diam } \sigma$. Then we can find disjoint l -simplices u_1, \dots, u_k in the interior of τ such that each u_i is isometric to s_i . Identify each s_i with u_i . After this there are no free faces contained in σ . Do this for every free face. After a finite number of steps we get a finite piecewise flat npc simplicial complex Y with no free faces. Consequently Y is geodesically complete. Also it is easy to see that $\pi_1(Y) = \pi_1(Z) * F_N$ for some large N . This proves proposition E.

proof of proposition 5. Let $n \geq 2$. Let $P \subset \mathbb{R}^2$ be a regular $4n$ -gon. We can identify the sides of P to obtain a piecewise flat npc 2-simplex Y homeomorphic to the surface of genus n . In this particular case the npc metric induced on Y by P is a flat metric, except for the vertex v , where the length ℓ of link is larger than 2π . By the PL Gauss-Bonnet formula $2\pi(2 - 2n) = 2\pi - \ell$. Thus $\ell = 2\pi(2n - 1) \geq 6\pi$. Let s_0 and s_1 be embedded segments on Y of the same length such that $s_0 \cap s_1 = v$ and the two angles determined by s_0 and s_1 at v are each larger than 2π . This is possible because $\ell \geq 6\pi$. Let Z be obtained from Y identifying s_0 with s_1 . This space Z satisfies the statement of the proposition. This proves proposition F.

Before proving proposition 3 we prove a lemma.

Lemma 4.1. *Let $a_1, \dots, a_j, \theta_1, \dots, \theta_j$ be positive real numbers. Then there is piecewise flat npc complex W homeomorphic to a 2-surface with an open disk deleted, such that the boundary of W consists of j geodesic segments s_1, \dots, s_j of lengths a_1, \dots, a_j and the angle at the initial point of s_i is larger than θ_i , $i = 1, \dots, j$.*

proof. Let Y be the space of the proof of proposition F. Let V be obtained from Y in

the following way. Delete the segment s_0 from Y and glue back two copies of s_0 which will intersect in their initial and final points. Then V is homeomorphic to a surface with an open disk deleted. Also V is piecewise flat and npc and its boundary consists of two segments and two vertices with angles 2π and $\ell = 2\pi(2n - 1)$ (see the proof of prop. F). Note that we can choose Y , n , and the length of s_0 arbitrarily. Hence taking several spaces like this V and gluing them along the boundaries it is not difficult to show that we can find the required W .

proof of Proposition 3. Let $m \geq 3$ be a positive integer. Let Δ^m be the canonical m -simplex. By proposition E there is a finite piecewise flat geodesically complete npc m -complex U with $\pi_1(U) = \pi_1(\Delta^m) * F_r = F_r$, for some large r , and we can assume r even. Let R be a 1-complex PL -embedded in U such that the inclusion $R \rightarrow U$ is a homotopy equivalence. Then R has the homotopy type of a wedge of r circles. Because r is even there is a map ρ from the boundary $\mathbb{S}^1 = \partial D$ of the 2-disk D to R such that if we glue D with the mapping cylinder of ρ along ∂D we obtain a closed surface of genus $\frac{r}{2}$. We can assume that ρ is a simplicial map and let s_1, \dots, s_j be a subdivision of ∂D in segments such that ρ is simplicial on each s_i . Let a_i be the length of $\rho(s_i)$ and choose $\theta_i \geq 2\pi - \alpha_i$, where α_i is the angle at the initial point of $\rho(s_i)$ between the segments $\rho(s_i)$ and $\rho(s_{i-1})$. Let W be the space given by lemma 8.1 corresponding to these a_i 's and θ_i 's. Hence we have a map $\psi : \partial W \rightarrow U$ with $\psi(\partial W) \subset R$, that is a local isometry. Let Z be obtained by gluing W and the mapping cylinder of $\psi : \partial W \rightarrow U$ along ∂W . Then Z is an m -complex and it is homotopically equivalent to a closed surface, and it is also a finite geodesically complete npc piecewise flat complex. This proves proposition 3.

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